AS and A-level Further Maths content

Version 1.1

The subject content for 50% of AS and A-level Further Mathematics is set out by the Department for Education (DfE) and is common across all exam boards. The remaining 50% could be chosen by each exam board. The content set out in this document covers the complete AS and A-level course of study for AQA.

Content required for AS Further Mathematics is shown in bold text. This, assessed in the context of the AS overarching themes, represents 100% of the AS content.

The content in standard type is assessed at A-level only.

A-level specifications in Further Mathematics must include the all of the following content in bold and standard type. This, assessed in the context of the overarching themes, represents 100% of the content.

A: Proof

A1

Construct proofs using mathematical induction; contexts include sums of series, divisibility, and powers of matrices.

B: Complex numbers

B1

Solve any quadratic equation with real coefficients; solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics).

B2

Add, subtract, multiply and divide complex numbers in the form $x + iy$ with x and y real; understand and use the terms 'real part' and 'imaginary part'.

B3

Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.

B4

Use and interpret Argand diagrams.

B5

Convert between the Cartesian form and the modulus-argument form of a complex number (knowledge of radians is assumed).

B6

Multiply and divide complex numbers in modulus-argument form (knowledge of radians and compound angle formulae is assumed).

B7

Construct and interpret simple loci in the Argand diagram such as $|z-a|<r$ and $\arg(z-a)=\theta$ (knowledge of radians is assumed).

B8

Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.

B9

Know and use the definition $e^{i\theta}=\cos\theta+i\sin\theta$ and the form $z=re^{i\theta}$

B10

Find the n distinct nth roots of $re^{i\theta}$ for $r\neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.

B11

Use complex roots of unity to solve geometric problems.

C: Matrices

C1

Add, subtract and multiply conformable matrices; multiply a matrix by a scalar.

C2

Understand and use zero and identity matrices.

C3

Use matrices to represent linear transformations in 2D; successive transformations; single transformations in 3D (3D transformations confined to reflection in one of $x=0$, $y=0$, $z=0$ or rotation about one of the coordinate axes) (knowledge of 3D vectors is assumed).

C4

Find invariant points and lines for a linear transformation.

C5

Calculate determinants of 2×2 matrices and 3×3 matrices and interpret as scale factors, including the effect on orientation.

C6

Understand and use singular and non-singular matrices; properties of inverse matrices.

Calculate and use the inverse of non-singular 2×2 matrices and 3×3 matrices.

C7

Solve three linear simultaneous equations in three variables by use of the inverse matrix.

C8

Interpret geometrically the solution and failure of solution of three simultaneous linear equations.

C9

Factorisation of determinants using row and column operations.

C10

Find eigenvalues and eigenvectors of 2×2 and 3×3 matrices.

Find and use the characteristic equation.

Understand the geometrical significance of eigenvalues and eigenvectors.

C11

Diagonalisation of matrices; $\mathbf{M}=\mathbf{UDU}^{-1}$; $\mathbf{M}^n=\mathbf{UD}^n\mathbf{U}^{-1}$ when eigenvalues are real.

D: Further algebra and functions.

D1

Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.

D2

Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).

D3

Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.

D4

Understand and use the method of differences for summation of series including use of partial fractions.

D5

Find the Maclaurin series of a function including the general term.

D6

Recognise and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$, $(1+x)^n$, and be aware of the range of values of x for which they are valid (proof not required).

D7

Evaluation of limits using Maclaurin series or l'Hôpital's rule.

D8

Inequalities involving polynomial equations (cubic and quartic).

D9

Solve inequalities such as $\frac{ax+b}{cx+d} < ex+f$ algebraically.

D10

Modulus of functions and associated inequalities.

D11

Graph of $y = |f(x)|$, $y = \frac{1}{f(x)}$ for given $y = f(x)$.

D12

Graphs of rational functions of form $\frac{ax+b}{cx+d}$; asymptotes, points of intersection with coordinate axes or other straight lines; associated inequalities.

D13

Graphs of rational functions of form $\frac{ax^2+bx+c}{dx^2+ex+f}$, including cases when some of these coefficients are zero; asymptotes parallel to coordinate axes; oblique asymptotes.

D14

Using quadratic theory (not calculus) to find the possible values of the function and coordinates of the stationary points of the graph for rational functions of form $\frac{ax^2+bx+c}{dx^2+ex+f}$

D15

Sketching graphs of curves with equation $y^2 = 4ax$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $xy = c^2$ including intercepts with axes and equations of asymptotes of hyperbolas.

D16

Single transformations of curves involving translations, stretches parallel to coordinate axes and reflections in the coordinate axes and the lines $y = \pm x$.

Extend to composite transformations including rotations and enlargements.

E: Further calculus

E1

Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.

E2

Derive formulae for and calculate volumes of revolution.

E3

Understand and evaluate the mean value of a function.

E4

Integrate using partial fractions (extend to quadratic factors $ax^2 + c$ in the denominator).

E5

Differentiate inverse trigonometric functions.

E6

Integrate functions of the form $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 + x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.

E7

Arc length and area of surface of revolution for curves expressed in Cartesian or parametric coordinates.

E8

Derivation and use of reduction formulae for integration.

E9

The limits $\lim_{x\to\infty}(x^k e^{-x})$ and $\lim_{x\to 0}(x^k \ln x)$ where $k > 0$, applied to improper integrals.

F: Further vectors

F1

Understand and use the vector and Cartesian forms of an equation of a straight line in 3D.

F2

Understand and use the vector and Cartesian forms of the equation of a plane.

F3

Calculate the scalar product and use it to calculate the angle between two lines, to express the equation of a plane, and to calculate the angle between two planes and the angle between a line and a plane.

F4

Check whether vectors are perpendicular by using the scalar product.

F5

Calculate and understand the properties of the vector product.

Understand and use the equation of a straight line in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$.

Use vector products to find the area of a triangle.

F6

Find the intersection of two lines.

Find the intersection of a line and a plane.

Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.

G: Polar coordinates

G1

Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.

G2

Sketch curves with *r* given as a function of θ, including use of trigonometric functions.

G3

Find the area enclosed by a polar curve.

H: Hyperbolic functions

H1

Understand the definitions of hyperbolic functions sinh x, cosh x and tanh x, including their domains and ranges, **and be able to sketch their graphs.**

Understand the definitions of hyperbolic functions sech x, cosech x and coth x, including their domains and ranges.

H2

Differentiate and integrate hyperbolic functions.

H3

Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

H4

Derive and use the logarithmic forms of the inverse hyperbolic functions.

H5

Integrate functions of the form $(x^2 + a^2)^{-\frac{1}{2}}$ and $(x^2 - a^2)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.

H6

Understand and use $\tanh x \equiv \dfrac{\sinh x}{\cosh x}$.

Understand and use $\cosh^2 x - \sinh^2 x \equiv 1$; $\operatorname{sech}^2 x \equiv 1 - \tanh^2 x$ and $\operatorname{cosech}^2 x \equiv \coth^2 x - 1$, $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$, $\sinh 2x \equiv 2\sinh x \cosh x$.

H7

Construct proofs involving hyperbolic functions and identities.

I: Differential equations

I1

Find and use an integrating factor to solve differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)$ and recognise when it is appropriate to do so.

I2

Find both general and particular solutions of differential equations.

I3

Use differential equations in modelling in kinematics and in other contexts.

I4

Solve differential equations of the form $y'' + ay' + by = 0$ where a and b are constants, by using the auxiliary equation.

I5

Solve differential equations of the form $y'' + ay' + by = \mathrm{f}(x)$ where a and b are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function).

I6

Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.

I7

Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion.

I8

Model damped oscillations using 2nd order differential equations and interpret their solutions.

Understand light, critical and heavy damping and be able to determine when each will occur.

I9

Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1st order simultaneous equations and be able to solve them, for example predator-prey models.

I10

Use of Hooke's Law with $T = kx$ to formulate a differential equation for simple harmonic motion, where k is a constant.

I11

Use models for damped motion where the damping force is proportional to the velocity.

J: Numerical methods

J1

Mid-ordinate rule and Simpson's rule for integration.

J2

Euler's step by step method for solving first order differential equations.

J3

Improved Euler method for solving first order differential equations.

$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$, $x_{r+1} = x_r + h$.

Optional application 1 – mechanics

MA: Dimensional analysis

MA1

Finding dimensions of quantities; checking for dimensional consistency.

MA2

Prediction of formulae; finding powers in potential formulae.

MB: Momentum and collisions

MB1

Conservation of momentum for linear motion and cases where velocities are given as one or two dimensional vectors (resolving will not be required at AS level, problems which require resolving will be required at A-level).

MB2

Coefficient of restitution and Newton's Experimental Law. Use in direct collisions and impacts with a fixed smooth surface (resolving will not be required at AS level, problems which require resolving will be required at A-level).

MB3

Impulse and its relation to momentum (in one- or two-dimensions) (resolving will not be required at AS level, problems which require resolving will be required at A-level).

Use of $Ft = mv - mu$ (resolving will not be required at AS level, problems which require resolving will be required at A-level).

MB4

Impulse for variable forces. One dimension only. Use of $I = \int F \, \mathrm{d}t$

MC: Work, energy and power

MC1

Work done by a force acting in the direction of motion or directly opposing the motion.

Use of $WD = Fd\cos\theta$.

MC2

Gravitational potential energy. Use in conservation of energy problems.

MC3

Kinetic energy. Use in conservation of energy problems.

MC4

Hooke's Law including using modulus of elasticity.

Use of $T = kx$ or $T = \frac{\lambda}{l}x$

MC5

Work done by a variable force.

Use of $WD = \int F \, \mathrm{d}x$.

Use in conservation of energy problems.

MC6

Elastic potential energy using modulus of elasticity. Use of $EPE = \frac{kx^2}{2}$ and $EPE = \frac{\lambda x^2}{2l}$.

Use in conservation of energy problems.

MC7

Power (resolving will not be required at AS level, problems which require resolving will be required at A-level).

Use of $P = Fv$.

MD: Circular motion

MD1

Motion of a particle moving in a circle with constant speed (knowledge of radians assumed).

MD2

Understand the definition of angular speed.

Use both radians and revolutions per unit time.

MD3

Relationships between speed, angular speed, radius and acceleration. Use of $v = r\omega$, $a = r\omega^2$ and $a = \frac{v^2}{r}$.

MD4

Use position, velocity and acceleration as vectors in the context of circular motion.

MD5

Conical pendulum, with one or two strings.

MD6

Circular motion in a vertical plane. Includes conditions to complete vertical circles. Use conservation of energy in this context.

ME: Centres of mass and moments

ME1

Centre of mass for a system of particles.

ME2

Centre of mass for a composite body.

ME3

Centre of mass of a lamina by integration.

ME4

Centres of mass of bodies formed by rotating a region about the x-axis.

ME5

Conditions for sliding and toppling. Problems including suspension and on an inclined plane.

ME6

Determine the forces acting on a rigid body in equilibrium. Use of moments and couples.

Optional application 2 – statistics

SA: Discrete random variables (DRVs) and expectation

SA1

Understand DRVs with distributions given in the form of a table or function.

SA2

Evaluate probabilities for a DRV.

SA3

Evaluate measures of average and spread for a DRV to include mean, variance, standard deviation, mode and median.

SA4

Understand expectation and know the formulae: $\mathbf{E}(X) = \sum x_i p_i$; $\mathbf{E}(X^2) = \sum x_i^2 p_i$; $\mathbf{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2$.

SA5

Understand expectation of linear functions of DRVs and know the formulae: $\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$ **and** $\mathbf{Var}(aX + b) = a^2\mathbf{Var}(X)$.

Know the formula $\mathrm{E}(\mathrm{g}(X)) = \sum \mathrm{g}(x_i) p_i$.

Find the mean, variance and standard deviation for functions of a DRV such as $\mathrm{E}(5(X^3))$, $\mathrm{E}(18X^{-3})$, $\mathrm{Var}(6X^{-1})$.

SA6

Know the discrete uniform distribution defined on the set $\{1,2,3,...,n\}$ **. Understand when this distribution can be used as a model.**

SA7

Proof of mean and variance of discrete uniform distribution.

SB: Poisson distribution

SB1

Understand conditions for a Poisson distribution to model a situation. Understand terminology $X \sim \text{Po}(\lambda)$.

SB2

Know the Poisson formula and calculate Poisson probabilities using the formula or equivalent calculator function.

SB3

Know mean, variance and standard deviation of a Poisson distribution.

Use the result that, if $X \sim \text{Po}(\lambda)$ then the mean and variance of X are equal.

SB4

Understand the distribution of the sum of independent Poisson distributions.

SB5

Formulate hypotheses and carry out a hypothesis test of a population mean from a single observation from a Poisson distribution using direct evaluation of Poisson probabilities.

SC: Type I and Type II errors

SC1

Understand Type I and Type II errors and define in context. Calculate the probability of making a Type I error from tests based on a Poisson or Binomial distribution.

Calculate probability of making Type I error from tests based on a normal distribution.

SC2

Understand the power of a test. Calculations of P(Type II error) and power for a test for tests based on a normal, Binomial or a Poisson distribution.

SD: Continuous random variables (CRVs)

SD1

Understand and use a probability density function, $\mathrm{f}(x)$, for a continuous distribution and understand the differences between discrete and continuous distributions.
Understand and use distributions of random variables that are part discrete and part continuous.

SD2

Find the probability of an observation lying in a specified interval.

SD3

Find the median and quartiles for a given probability density function, $\mathrm{f}(x)$.

SD4

Find the mean, variance and standard deviation for a given pdf, $\mathrm{f}(x)$. Know the formula $\mathrm{E}(X) = \int x\mathrm{f}(x)\,\mathrm{d}x$, $\mathrm{E}(X^2) = \int x^2\mathrm{f}(x)\,\mathrm{d}x$, $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$.

SD5

Understand the expectation and variance of linear functions of CRVs and know the formulae:

$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

Know the formula $\mathrm{E}(\mathrm{g}(X)) = \int \mathrm{g}(x)\mathrm{f}(x)\,\mathrm{d}x$.

Find the mean, variance and standard deviation of functions of a continuous random variable such as $\mathrm{E}(5X^3)$, $\mathrm{E}(18X^{-3})$, $\mathrm{Var}(6X^{-1})$.

SD6

Understand and use a cumulative distribution function, $\mathrm{F}(x)$. Known the relationship between $\mathrm{f}(x)$ and $\mathrm{F}(x)$.

$$\mathrm{F}(x) = \int_{-\infty}^{x} \mathrm{f}(t)\,\mathrm{d}t \text{ and } \mathrm{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{F}(x).$$

SD7

Understand the rectangular distribution $\mathrm{f}(x)$ where

$$\mathrm{f}(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Know the conditions for the rectangular distribution to be used as a model.

Calculate probabilities from a rectangular distribution.

Know proofs of mean, variance and standard deviation for a rectangular distribution.

SD8

Know that if X and Y are independent (discrete or continuous) random variables then E(X + Y) = E(X) + E(Y) and Var(X + Y) = Var(X) + Var(Y).

SE: Chi squared tests for association

SE1

Construction of $n \times m$ contingency tables.

SE2

Use of $\sum \frac{(O_i - E_i)^2}{E_i}$ as an appropriate χ^2 statistic with appropriate degrees of freedom.

SE3

Know and use the convention that all E_i should be greater than 5.

SE4

Identification of sources of association in the context of a question.

SE5

Knowledge of when and how to apply Yates' correction.

SF: Exponential distribution

SF1

Know the conditions for an exponential distribution to be used as a model. Know the probability density function, $\mathrm{f}(x)$, and the cumulative distribution function, $\mathrm{F}(x)$, for an exponential distribution.

SF2

Calculate probabilities for an exponential distribution using $\mathrm{F}(x)$ or integration of $\mathrm{f}(x)$.

SF3

Know proofs of mean, variance and standard deviation for an exponential distribution.

SF4

Understand that the length of intervals between Poisson events have an exponential distribution.

SG: Inference – one sample *t*-distribution

SG1

Test for the mean of a normal distribution with unknown variance using a *t*-statistic with appropriate degrees of freedom.

SH: Confidence intervals

SH1

Construct symmetric confidence intervals for the mean of a normal distribution with known variance.

SH2

Construct symmetric confidence intervals from large samples, for the mean of a normal distribution with unknown variance.

SH3

Make inferences from constructed or given confidence intervals.

SH4

Construct symmetric confidence intervals from small samples, for the mean of a normal distribution with unknown variance using the *t*-distribution.

Optional application 3 – discrete mathematics

DA: Graphs

DA1

Understand and use the language of graphs including: vertex, edge, trail, cycle, connected, degree, subgraph, subdivision, multiple edge and loop.

DA2

Identify or prove properties of a graph including that a graph is Eulerian, semi-Eulerian or Hamiltonian.

DA3

Understand and use Euler's formula for connected planar graphs.

DA4

Use Kuratowski's Theorem to determine the planarity of graphs.

DA5

Understand and use complete graphs and bipartite graphs, including adjacency matrices and the complement of a graph.

DA6

Understand and use simple graphs, simple-connected graphs and trees.

DA7

Recognise and find isomorphism between graphs.

DB: Networks

DB1

Understand and use the language of networks including: node, arc and weight.

DB2

Solve network optimisation problems using spanning trees.

DB3

Solve route inspection problems.

DB4

Find and interpret upper bounds and lower bounds for the travelling salesperson problem.

DB5

Evaluate, modify and refine models which use networks.

DC: Network flows

DC1

Interpret flow problems represented by a network of directed arcs.

DC2

Find the value of a cut and understand its meaning.

DC3

Use and interpret the maximum flow-minimum cut theorem.

DC4

Introduce supersources and supersinks to a network.

DC5

Augment flows and determine the maximum flow in a network.

DC6

Solve problems involving arcs with upper and lower capacities.

DC7

Refine network flow problems including using nodes of restricted capacity.

DD: Linear programming

DD1

Formulate constrained optimisation problems.

DD2

Solve constrained optimisation problems via graphical methods.

DD3

Use the Simplex algorithm for optimising (maximising and minimising) an objective function including the use of slack variables.

DD4

Interpret a Simplex tableau.

DE: Critical path analysis

DE1

Construct, represent and interpret a precedence (activity) network using activity-on-node.

DE2

Determine earliest and latest start and finish times for an activity network.

DE3

Identify critical activities, critical paths and the float of non-critical activities.

DE4

Refine models and understand the implications of possible changes in the context of critical path analysis.

DE5

Construct and interpret Gantt (cascade) diagrams and resource histograms.

DE6

Carry out resource levelling (using heuristic procedures) and solve problems where resources are restricted.

DF: Game theory for zero-sum games

DF1

Understand, interpret and construct pay-off matrices.

DF2

Find play-safe strategies and the value of the game.

DF3

Prove the existence or non-existence of a stable solution.

DF4

Identify and make use of dominated strategies.

DF5

Find optimal mixed strategies for a game including use of graphical methods.

DF6

Convert and solve higher order games to linear programming problems.

DG: Binary operations

DG1

Understand and use binary operations including use of modular arithmetic and matrix multiplication.

DG2

Understand, use and prove the commutativity of a binary operation.

DG3

Understand, use and prove the associativity of a binary operation.

DG4

Construct a Cayley table for a given set under a given binary operation.

DG5

Understand and prove the existence of an identity element for a given set under a given binary operation.

DG6

Find the inverse of an element belonging to a given set under a given binary operation.

DG7

Understand and use the language of groups including: order, period, subgroup, proper, trivial, non-trivial.

DG8

Understand and use the group axioms: closure, identity, inverses and associativity, including use of Cayley tables.

DG9

Recognise and use finite and infinite groups and their subgroups, including: groups of symmetries of regular polygons, cyclic groups and abelian groups.

DG10

Understand and use Lagrange's theorem.

DG11

Identify and use the generators of a group.

DG12

Recognise and find isomorphism between groups of finite order.

Overarching themes

AS and A-level specifications in further mathematics must require students to demonstrate the following overarching knowledge and skills. These must be applied, along with associated mathematical thinking and understanding, across the whole of the detailed content set out above.

OT1: Mathematical argument, language and proof

OT1.1

Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable.

OT1.2

Understand and use mathematical language and syntax as set out in the content.

OT1.3

Understand and use language and symbols associated with set theory, as set out in the content.

OT1.4

Understand and use the definition of a function; domain and range of functions.

OT1.5

Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics.

OT2: Mathematical problem solving

OT2.1

Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved.

OT2.2

Construct extended arguments to solve problems presented in an unstructured form, including problems in context.

OT2.3

Interpret and communicate solutions in the context of the original problem.

OT2.6

Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.

OT2.7

Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems, including in mechanics.

OT3: Mathematical modelling

OT3.1

Translate a situation in context into a mathematical model, making simplifying assumptions.

OT3.2

Use a mathematical model with suitable inputs to engage with and explore situations (for a given model or a model constructed or selected by the student).

OT3.3

Interpret the outputs of a mathematical model in the context of the original situation (for a given model or a model constructed or selected by the student).

OT3.4

Understand that a mathematical model can be refined by considering its outputs and simplifying assumptions; evaluate whether the model is appropriate.

OT3.5

Understand and use modelling assumptions.